Gamma Function and Related with another Functions

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***Abstract*----**The gamma function (factorial function) appears occasionally in physical problem such as the normalization of coulomb wave function and the computation of probabilities in statistical machines. In general, however it has less direct physical application and interpretation.

***Keywords*---** Gamma, related, another function

I. Introduction

IN the recent years of this century, applications of special functions as solutions of physical differential equations, and integral equations have expanded from here the importance of research related to systems, Gamma function and related with another special functions.

A. Definitions

At Least three different Convenient definitions of the gamma function are in common use. Our first task is to state these definitions, to develop some simple direct consequences, and to show the equivalence of the three forms.

1) The gamma function was first defined in 1729 by the great Swiss mathematician Euler. He defined the Gamma function by an Infinite product[1,2]:

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{z}}{\left(1+\frac{z}{n}\right)} \dots\dots\dots\dots (1)$$

If z be taken as the complex variable $x + iy$ Euler's product der converges at every finite z except $z = 0, -1, -2 \dots$

The function defined by the product in analytic at every finite z except for the singular points just mentioned. At each of the singular points, $\Gamma(z)$ has a simple pole

2**)** Infinite limit (Euler)[1,3].

$$\Gamma(z) = \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\dots\dots\dots . n}{(z+1)(z+2)\dots\dots\dots\dots(z+n)}n^{z} \dots\dots\dots (2)$$

Hare z may be either real or complex replacing z with (z+1), we have:

$$\Gamma(z+1) = \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\dots\dots n}{(z+1)(z+2)(z+3)\dots\dots(z+n+1)}n^{z+1} \dots . (\acute{2})$$

$$= \lim_{n\to\infty}\frac{nz}{(z+n+1)}\cdot\frac{1\cdot 2\cdot 3\dots\dots n}{z(z+1)(z+2)\dots\dots(z+n)}n^{z}$$

$$= z\,\Gamma(z)$$

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This is the basic functional relation for the Gamma functions. It should be noted that it is a difference equation. It has been shown that the Gamma function in one of a general class of functions which do not satisfy any differential equation with rational coefficients. Also from the definition:

$$\Gamma(1) = \lim_{n\to\infty}\frac{1\cdot 2\cdot 3\dots\dots\dots n\cdot n}{1\cdot 2\cdot 3\dots\dots\dots\dots n(n+1)} = 1$$

The notation $\Gamma(z)$ and the name "Gamma function" were first used by Legendre in 1814.

3**)** From Euler's infinite product for $\Gamma(z)$ can be derived the formula[3]:

$$\Gamma(z) = \int_{0}^{\infty} e^{-t}\,t^{z-1}dt \dots (3)$$

This integral formula is convergent only when the real part of z is positive. Never the less this integral formula for Γ(z) often is taken as the starting point for introductory treatments of the Gamma function. Moreover the variable z is often contrary is explicitly stated, we shall be concerned in our exercises and problems with the Gamma function of a real variable only. For positive values of x we shall take the following as our basic definition of the Gamma function:

$$\Gamma(x) = \int_{0}^{\infty} e^{x-1}\,e^{-t}dt\,;\, x > 0 \dots\dots\dots\dots\dots (\acute{3})$$

As is usually done, we shall extend the domain of the gamma function into the realm of negative number (exclusive of negative integers) by extra potation via the characteristic equation.

$$\Gamma(x+1) = x\Gamma(x)$$

B. The Gamma function and related functions

In developing series solutions of differential equations and in other formal calculations it is often convenient to make use of properties of gamma and beta function. The integral[4,5]:

$$\Gamma(n) = \int_{0}^{\infty} e^{-x}\,x^{n-1}dx \dots\dots\dots\dots\dots\dots\dots\dots\dots (\acute{\acute{3}})$$

Converges if n > ∞ and defines the gamma function.

Similarly if m > 0, n > 0 the beta function is defined by equation:

$$\frac{\Gamma m\Gamma n}{\Gamma(m+n)} = \beta(\min) = \int_{0}^{1} x^{m-1}\,(1-x)^{n-1}dx$$

C. Results

It is then easily shown that

i. $\Gamma(1) = 1$

ii. $\Gamma(n+1) = n\Gamma(n)$

iii. $\Gamma(n+1) = n!\,;$ if n is a positive integer

iv. $$\frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \beta(\min) = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta \sin^{2m-1}\theta\, d\theta$$

v. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

vi. $\Gamma(p)\,\Gamma(1-p) = \pi cose(p\pi);\qquad 0 < p < 1$

vii. $\Gamma\left(\frac{1}{2}\right)\Gamma(2n) = 2^{2n-1}\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right);$ —the duplication formula

viii. $$\Gamma(z+1) = \lim_{n\to\infty}\frac{n!\,n^2}{(z+1)(z+2)\ldots\ldots(z+n)};\ z > 0$$

When n is a negative fraction $\Gamma(n)$ is defined by means of equation $\Gamma(n+1) = n\Gamma(n)$.

For example:

$$\Gamma\left(\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)} = \frac{4\Gamma\left(\frac{1}{2}\right)}{3}$$

By means of the result (viii).

We can derive an interesting expression for Euler's constant, γ which is defined by the equation:

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) = 0.5772$$

From (viii) we have:

$$\frac{d}{dz}(\log\Gamma(z+1)) = \lim_{n\to\infty}\left(\log n - \frac{1}{z+1} - \frac{1}{z+2} - \cdots\frac{1}{z+n}\right)$$

So that letting $z \to 0$ we obtain the result.

$$\gamma = -\left[\frac{d}{dz}\log\Gamma(z+1)\right]_{z=2}$$

And from (2) we find

$$\gamma = -\int_0^\infty e^{-t}\log t\, dt$$

Integrating by parts, we see that:

$$\int_0^\infty e^{-t}\log t\, dt = e^{-z}\log z + \int_z^\infty \frac{e^{-t}}{t}dt$$

so that:

$$-\gamma = \lim_{z\to 0}\left(\int_0^\infty \frac{e^{-t}}{t}dt + \log 2\right)$$

Closely related to the gamma function are the exponential-integral ei(x) defined by the equation.

$$\mathrm{ei}(x) = \int_0^\infty \frac{e^{-u}}{u}du;\quad x > 0$$

And the logarithmic-integral Li(x) defined by:

$$\mathrm{Li}(x) = \int_0^\infty \frac{du}{\log u}$$

Which are themselves connected by the relation:

$$\mathrm{ei}(x) = \mathrm{Li}(e^{-x})$$

Other integrals of importance are the sine and cosine integrals Ci(x), Si(x) which are defined by the equations:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin u}{u}du$$

And

$$\mathrm{Ci}(x) = \int_0^\infty \frac{\cos u}{u}du$$

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